JOURNAL OF MODERN OPTICS, 1991, VOL. 38, NO. 5, 857–864

Exact solutions of laser rate equations

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(*Received 12 January 1990; revision received 31 August 1990*)

Abstract. An analytical analysis of the rate equations for three- and four-level laser systems is presented. Exact closed-form analytical relationships are derived between the inversion population density $N(t)$, the photon number $q(t)$ and the time t for specific values of the pump rate W_p. Among the various solutions that are presented, the general analytical relationship between $q(t)$ and $N(t)$ for a three-level system at the threshold pump rate is derived.

1. Introduction

There are, in general, three approaches to laser theory: quantum theory, semiclassical theory and rate equation theory. By treating both the atoms and the electromagnetic fields quantum-mechanically, the quantum theory is able to describe the behaviour of laser systems more accurately than are the other two theoretical approaches. The pitfall of the quantum theory is its complexity; solution of the resulting equations often requires sophisticated numerical procedures and considerable central processing unit time. The quantum approach is rarely used for applications other than studying phenomena that are not predicted by the other two theories (e.g. statistical behaviour and noise). Semiclassical theory treats the electromagnetic fields classically, while the atoms are treated quantum-mechanically. This approach properly accounts for coherence and multimode operation of laser systems. The rate equation theory is the limiting case of semiclassical theory for short coherence decay times. It is based upon simple conservation arguments and, while it does not treat coherence properly, it is successful in describing a variety of phenomena including threshold behaviour, spiking, continuous-wave operation and Q-switched operation. The rate equation approach is the method that is considered in this paper.

Two excellent reviews of the rate equation theory have been presented by Siegman [1] and Svelto [2] (Svelto's notation was chosen for use in this paper). Consider figure 1 where the energy levels of a three-level laser are schematically depicted. If it is assumed that the inversion population density and the number of photons within the laser cavity are spatially uniform, and that the laser operates with one dominant mode, then the application of basic conservation principles results in the following set of coupled nonlinear ordinary differential equations which govern the behaviour of this system:

$$\frac{dq}{dt}=V_a BqN-\frac{1}{\tau_c}q, \qquad q(t_0)=q_0, \tag{1}$$

$$\frac{dN}{dt}=-2BqN+W_p(N_t-N)-\frac{1}{\tau}(N_t+N), \qquad N(t_0)=N_0. \tag{2}$$

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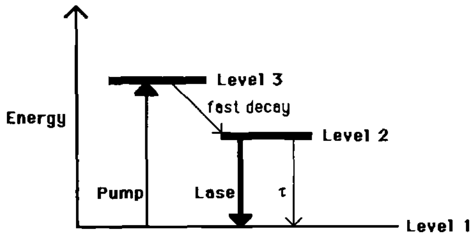

Figure 1. Atomic energy levels and transitions in a three-level laser. Atoms are pumped from the ground state (level 1) to an excited energy state (level 3) and then quickly decay, relative to the spontaneous decay rate from level 2 to level 1, to a lower-energy state (level 2). The decay from level 2 to the ground state consists of both stimulated emission (Lase) and spontaneous emission (τ).

Definitions of parameters appearing in equations (1)–(4) of the text.

Parameter	Definition	Units
t	Time	(s)
q	Number of photons	(—)
N	Inversion population density $= N_2 - N_1$	(cm^{-3})
N_t	Total population density $= N_1 + N_2$ for a three-level system $= N_g + N_2$ for a four-level system	(cm^{-3})
N_2	Population density of energy level 2	(cm^{-3})
N_1	Population density of energy level 1	(cm^{-3})
N_g	Ground-level population density	(cm^{-3})
V_a	Mode volume in active medium	(cm^3)
V	Volume of laser cavity	(cm^3)
τ	Spontaneous emission lifetime	(s)
τ_c	Cavity lifetime	(s)
B	Transition rate per photon $= \sigma c V^{-1}$	(s^{-1})
σ	Stimulated transition cross section	(cm^2)
c	Speed of light	($cm\,s^{-1}$)
W_p	Pump rate	(s^{-1})

These equations are known as the three-level rate equations. Each parameter appearing in equations (1) and (2) is defined in the table, followed by appropriate units in parentheses. Equation (1) states that the rate (dq/dt) at which the number of photons within the laser cavity changes with respect to time is equal to the rate (V_aBqN) at which photons are produced within the active medium by stimulated emission minus the rate ($q\tau_c^{-1}$) at which photons leave the laser cavity through the end mirrors. Equation (2) may be interpreted in an analogous manner; the rate (dN/dt) at which the inversion population density changes with respect to time is equal to the rate ($-2BqN$) at which the inversion decays due to stimulated emission plus the rate ($W_pN_t - W_pN$) at which the ground-level atoms are pumped back up to the excited state minus the rate ($(N_t + N)\tau - 1$) at which the upper level (level 2 in figure 1) decays owing to spontaneous emission. Consider figure 2 where a four-level system is schematically depicted. The rate equations for this system are

$$\frac{dq}{dt} = V_aBqN - \frac{1}{\tau_c}q, \qquad q(t_0) = q_0, \tag{3}$$

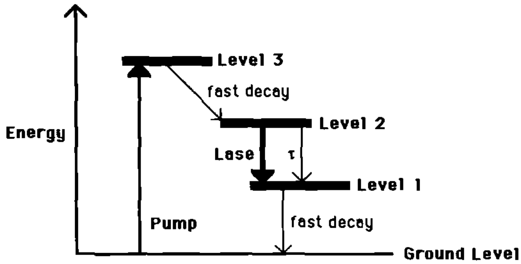

Figure 2. Atomic energy levels and transitions in a four-level laser. Atoms are pumped from the ground level to an excited energy state (level 3) and then quickly decay, relative to the spontaneous decay rate from level 2 to level 1, to a lower-energy state (level 2). The decay from level 2 to level 1 consists of both stimulated emission (Lase) and spontaneous emission (τ).

$$\frac{\mathrm{d}N}{\mathrm{d}t} = -BqN + W_p(N_t - N) - \frac{1}{\tau}N, \qquad N(t_0) = N_0. \tag{4}$$

While the rate equations have been used successfully in describing many of the basic characteristics of laser systems, exact closed-form analytical solutions have not been determined. With the exception of special limiting cases such as steady-state operation, these equations must be integrated numerically. However, even standard numerical procedures for solving coupled differential equations must be applied with great care in determining transient solutions to the rate equations because these equations are nonlinear, and the physical constants in the equations typically differ by several orders of magnitude.

Researchers have attempted to determine analytical solutions to these equations but have gained only limited success. By assuming that the inversion population density and photon number could each be expanded in an infinite power series in time, Marchi, Millán and Crinó [3] were able to solve the three-level equations. Unfortunately, the coefficients in these solutions are complicated functions of the physical constants, and it seems likely that the determination of numerical values from these infinite-power-series expansions would present more difficulty than solving the original equations directly by standard numerical procedures.

The determination of analytical solutions would provide researchers with information on how the inversion population density and photon number depend upon time and upon the physical constants of the laser systems that these equations model. It is the purpose of this paper to derive exact closed-form analytical solutions to the rate equations. The methods of solution are described in the following section.

2. Analysis

The equations for both three-level and four-level systems may be represented by

$$\frac{\mathrm{d}q}{\mathrm{d}t} = V_a BqN - \frac{1}{\tau_c}q, \qquad q(t_0) = q_0, \tag{5}$$

$$\frac{\mathrm{d}N}{\mathrm{d}t} = c_1 qN - \left(W_p + \frac{1}{\tau}\right)N + c_2, \qquad N(t_0) = N_0, \tag{6}$$

where

$$c_1 = -2B, \qquad c_2 = N_t\left(W_p - \frac{1}{\tau}\right), \tag{7}$$

for a three-level system, and

$$c_1 = -B, \qquad c_2 = N_t W_p, \tag{8}$$

for a four-level system. Solving for qN from both (5) and (6) and setting the results equal,

$$\frac{1}{V_a B}\left(\frac{dq}{dt} + \frac{1}{\tau_c} q\right) = \frac{1}{c_1}\left[\frac{dN}{dt} + \left(W_p + \frac{1}{\tau}\right)N - c_2\right]. \tag{9}$$

Equation (9) may be equivalently expressed as

$$\exp\left(-\frac{t}{\tau_c}\right)\frac{d}{dt}\left[q \exp\left(\frac{t}{\tau_c}\right)\right] = \frac{V_a B}{c_1} \exp\left[-\left(W_p + \frac{1}{\tau}\right)t\right] \times \frac{d}{dt}\left\{N \exp\left[\left(W_p + \frac{1}{\tau}\right)t\right]\right\} - \frac{V_a B c_2}{c_1}. \tag{10}$$

For simplicity, suppose first that $W_p + \tau^{-1} = \tau_c^{-1}$, for which equation (10) becomes

$$\frac{d}{dt}\left[\exp\left(\frac{t}{\tau_c}\right)\left(q - \frac{V_a B}{c_1} N\right)\right] = -\frac{V_a B c_2}{c_1} \exp\left(\frac{t}{\tau_c}\right). \tag{11}$$

Integrating (11) with respect to time gives

$$N = \frac{c_1}{V_a B} q + c_2 \tau_c + C \exp\left(-\frac{t}{\tau_c}\right). \tag{12}$$

The parameter C in equation (12) is a constant of integration. Substituting for N from (12) into (5) gives

$$\frac{dq}{dt} = c_1 q^2 + \left[V_a B c_2 \tau_c - \frac{1}{\tau_c} + V_a B C \exp\left(-\frac{t}{\tau_c}\right)\right] q. \tag{13}$$

This is a Bernoulli equation. The solution to (13) is

$$q(t) = -\left\{\eta \exp\left[-u(t)\right] + c_1 \exp\left[-u(t)\right] \int \exp\left[u(t)\right] dt\right\}^{-1}, \tag{14}$$

where η is an integration constant and

$$u(t) = \left(V_a B c_2 \tau_c - \frac{1}{\tau_c}\right)t - V_a B C \tau_c \exp\left(-\frac{t}{\tau_c}\right). \tag{15}$$

Substituting (14) into (12) gives

$$N(t) = -\frac{c_1}{V_a B}\left\{\eta \exp\left[-u(t)\right] + c_1 \exp\left[-u(t)\right] \int \exp\left[u(t)\right] dt\right\}^{-1} + c_2 \tau_c + C \exp\left(-\frac{t}{\tau_c}\right). \tag{16}$$

Equations (14)–(16) are the time-dependent solutions of the rate equations for both three- and four-level systems for a pump rate W_p of $\tau_c^{-1} - \tau^{-1}$.

In order for a laser to reach threshold, a critical pump rate is required. If $N_t \gg N_c$ where $N_c = (V_a B \tau_c)^{-1}$ is the saturation value of the inversion population density, then the pump rate necessary for a three-level system to reach the lasing threshold is given by $W_p = \tau^{-1}$, while for a four-level system it is given by $W_p = N_c(N_t \tau)^{-1}$. For near-threshold operation, the pump rate that was assumed in deriving equations (14) and (16) limits the application of these solutions to systems where τ and τ_c are of the same order of magnitude. With the exception of the special cases where $\tau=\tau_c$ for a four-level system, or $\tau_c \leqslant \tau \leqslant 2\tau_c$ for a three-level system, the integral which appears in (14) and (16) cannot be solved exactly in closed analytical form for physically meaningful values of the constants. In most cases, the integrations must be performed numerically.

For the special case $c_2=0$, the integral in (14) and (16) becomes

$$\int \exp[u(t)]\,dt = \frac{1}{V_a BC}\exp\left[-V_a BC\tau_c \exp\left(-\frac{t}{\tau_c}\right)\right]+\beta \tag{17}$$

where β is an integration constant. Substituting (17) into both (14) and (16) and applying the initial conditions $N(t=0)=N_0$, $q(t=0)=q_0$, we obtain

$$q(t)=\left[\left(\frac{1}{q_0}+\frac{c_1}{V_a BC}\right)\exp\left\{\frac{t}{\tau_c}+V_a BC\tau_c\left[\exp\left(-\frac{t}{\tau_c}\right)-1\right]\right\}-\frac{c_1}{V_a BC}\exp\left(\frac{t}{\tau_c}\right)\right]^{-1}, \tag{18}$$

$$N(t)=\left[\left(\frac{V_a B}{c_1 q_0}+\frac{1}{C}\right)\exp\left\{\frac{t}{\tau_c}+V_a BC\tau_c\left[\exp\left(-\frac{t}{\tau_c}-1\right)-1\right]\right\}\right.$$
$$\left.-\frac{1}{C}\exp\left(\frac{t}{\tau_c}\right)\right]^{-1}+C\exp\left(-\frac{t}{\tau_c}\right). \tag{19}$$

The constant C in (18) and (19) is given by

$$C=N_0-\frac{c_1}{V_a B}q_0. \tag{20}$$

From both (7) and (8) it may be seen that the condition $c_2=0$ corresponds to a pump rate of $W_p=\tau^{-1}$ for a three-level system, or to a pump rate of $W_p=0$ in a four-level system. Equating these pump rates with the pump rate that was assumed in deriving equation (11), $W_p=(\tau_c^{-1}-\tau^{-1})$, it is clear that (18) and (19) are valid only if $\tau=2\tau_c$ for a three-level system, or $\tau=\tau_c$ for a four-level system. Equations (18) and (19) are therefore the analytical solutions to the rate equations at the threshold pump rate in a three-level system where $\tau=2\tau_c$.

A straightforward application of the methods of calculus reveal that the solution $q(t)$ given by (18) has at most two critical points (except the trivial solution $q(t)=0$ in which case every point is a critical point). One of these points, $q=0$, $t=\infty$, is an absolute minimum. The other critical point is an absolute maximum which occurs at the value of time implicitly defined by

$$0=-1+\left(\frac{V_a BC\tau_c}{c_1 q_0}+\tau_c\right)\left[\frac{1}{\tau_c}-V_a BC\exp\left(-\frac{t}{\tau_c}\right)\right]\exp\left\{V_a BC\tau_c\left[\exp\left(-\frac{t}{\tau_c}\right)-1\right]\right\}. \tag{21}$$

The maximum photon number is found by substituting the value for t from equation (21) into equation (18).

Equations (18) and (19) may be combined to implicitly define $q(t)$ as a function of $N(t)$. The result is

$$0=\frac{1}{\tau_c}\ln\left(\frac{N}{N_0}\right)-\left(W_p+\frac{1}{\tau}\right)\ln\left(\frac{q}{q_0}\right)-V_a B(N-N_0)+c_1(q-q_0). \tag{22}$$

Although equation (22) was derived under the assumption that $\tau = 2\tau_c$ for a three-level system, or $\tau = \tau_c$ for a four-level system, this equation is valid even if these assumptions are removed. The only assumptions that need to be made to derive (22) are that $c_2 = 0$, $q > 0$ and $N \neq 0$. This may be shown in the following manner.

Integrating equation (9) with respect to time (and leaving the integration constants within the unsolved integrals), we obtain

$$q + \frac{1}{\tau_c}\int q\,\mathrm{d}t = \frac{V_a B}{c_1}N + \frac{V_a B}{c_1}\left(W_p + \frac{1}{\tau}\right)\int N\,\mathrm{d}t - \frac{V_a B c_2}{c_1}t. \tag{23}$$

Rearranging (5) gives

$$\frac{1}{q}\frac{\mathrm{d}q}{\mathrm{d}t} = V_a B N - \frac{1}{\tau_c}. \tag{24}$$

Integrating (24) with respect to time (and, once again, leaving the integration constant within the unsolved integral), we find that

$$V_a B \int N\,\mathrm{d}t = \ln q + \frac{t}{\tau_c}. \tag{25}$$

Substituting (25) into (23) and rearranging gives

$$\frac{1}{\tau_c}\int q\,\mathrm{d}t = -q + \frac{V_a B}{c_1}N + \frac{W_p + 1/\tau}{c_1}\ln q + \left(\frac{W_p + 1/\tau}{c_1 \tau_c} - \frac{V_a B c_2}{c_1}\right)t. \tag{26}$$

Dividing equation (6) by N (assuming that $N \neq 0$), integrating and rearranging, we obtain

$$\frac{1}{\tau_c}\int q\,\mathrm{d}t = \frac{1}{c_1 \tau_c}\left[\ln N + \left(W_p + \frac{1}{\tau}\right)t - c_2\int \frac{1}{N}\mathrm{d}t + \tau_c K\right]. \tag{27}$$

The term $\tau_c K$ in (27) is an integration constant. Substituting (27) into (26) and rearranging give

$$\frac{c_2}{\tau_c}\int \frac{1}{N}\mathrm{d}t = \frac{1}{\tau_c}\ln N - \left(W_p + \frac{1}{\tau}\right)\ln q - V_a B N + c_1 q + V_a B c_2 t + K. \tag{28}$$

If $c_2 = 0$, then equation (28) becomes

$$0 = \frac{1}{\tau_c}\ln N - \left(W_p + \frac{1}{\tau}\right)\ln q - V_a B N + c_1 q + K. \tag{29}$$

Applying the initial condition $q(t_0) = q_0$, $N(t_0) = N_0$, equation (29) becomes

$$0 = \frac{1}{\tau_c}\ln\left(\frac{N}{N_0}\right) - \left(W_p + \frac{1}{\tau}\right)\ln\left(\frac{q}{q_0}\right) - V_a B(N - N_0) + c_1(q - q_0). \tag{30}$$

Inspection shows that equations (22) and (30) are identical. Equation (30) is the general analytical relationship between $q(t)$ and $N(t)$ at the threshold pump rate in a three-level system, or at a pump rate of $W_p = 0$ in a four-level system.

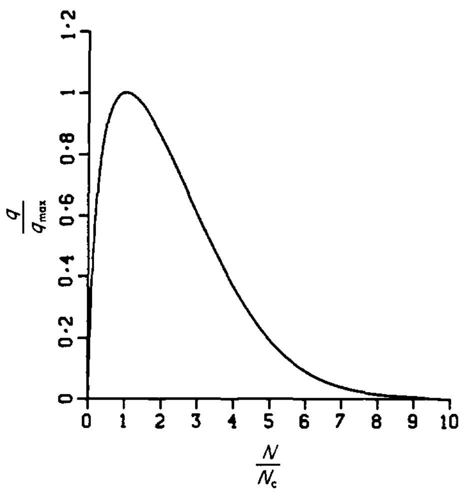

Figure 3. Graphical representation of equation (30) for a ruby laser at the threshold pump rate.

If equation (30) is interpreted as defining q as a function of N, then a straightforward application of the methods of calculus reveals that q has exactly two critical points. Dividing equation (5) by equation (6), we find that

$$\frac{dq}{dN}=\frac{V_a BqN-q/\tau_c}{c_1 qN-(W_p+1/\tau)N}. \tag{31}$$

Setting $dq/dN=0$ results in two critical points. One of these points, $q=0$, $N=\infty$, cannot, for obvious reasons, be observed physically. The other critical point is an absolute maximum which occurs at $N=N_c$ and the corresponding value of q is implicitly defined by

$$0=-\frac{1}{\tau_c}\ln(V_a B\tau_c N_0)-\left(W_p+\frac{1}{\tau}\right)\ln\left(\frac{q_{max}}{q_0}\right)+\left(V_a BN_0-\frac{1}{\tau_c}\right)+c_1(q_{max}-q_0), \tag{32}$$

where the value attained by q is denoted q_{max}. A characteristic graphical representation of (30) is given in figure 3, where normalized q is plotted against normalized N. The behaviour of equation (30) is consistent with the results of the functional analysis provided here.

3. Conclusions

An analytical analysis of the rate equations for three- and four-level laser systems has yielded exact analytical relationships between the inversion population density $N(t)$, the photon number $q(t)$ and the time t for specific values of the pump rate W_p. Equation (12) gives the analytical relationship between N, q and t for a pump rate of $W_p=\tau_c^{-1}-\tau^{-1}$ for both three- and four-level systems, and (14)–(16) give the explicit time dependences of both q and N for this pump rate. Equations (18) and (19) are the exact closed-form analytical solutions to the rate equations at the threshold pump rate in a three-level system where $\tau=2\tau_c$, and these equations also represent the solution for a pump rate of $W_p=0$ in a four-level system where $\tau=\tau_c$. Equation (30) gives the general analytical relationship between q and N for a pump rate W_p of τ^{-1} in a three-level system, or $W_p=0$ in a four-level system.

4. Discussion

In the past, the rate equations were solved numerically to determine the behaviour of a given laser system. With the exception of special limiting cases, analytical solutions had not been determined, and so numerical integration was the only alternative. Now that analytical solutions have been obtained, researchers have a new means of judging the accuracy and limitations of the rate equation approach. The closed-form analytical relationships that have been derived in this paper are, to the author's knowledge, the only exact solutions to the rate equations that have ever been found.

Until now, the examination of transient solutions could only be conducted with the aid of rather sophisticated numerical procedures. Equations (18) and (19) demonstrate very clearly why this is so; some terms in these equations decay at the rate of an exponential raised to the power of a decaying exponential. Although the more general class of solutions given by (14) and (16) must be solved numerically, the algorithms that are used to accomplish this task are relatively simple. For a three-level system at the threshold pump rate, equation (30) may be solved in conjunction with either equation (5) or equation (6) to determine transient solutions, thus reducing the problem of solving two coupled nonlinear differential equations to one of solving an algebraic equation and a nonlinear differential equation. At the very least, these analytical relationships reduce the complexity of the algorithms that are needed to determine numerical solutions.

In summary, the solutions presented in this paper represent the only exact closed-form analytical solutions to the rate equations that are known. Experimental work is required to determine the accuracy and limitations of these solutions, and it is hoped that the results presented in this paper will motivate future investigators to further the topic of laser theory.

Acknowledgments

Thanks are due to both Dr Carlos Stroud and Dr Mark Skeldon for their valuable input, and to Dr David Quesnel for his continuing support.

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