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Gravitational analysis of V541 Cygni, DI Herculis, and the Pioneer anomaly

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Abstract Detailed analyses by independent research groups over several decades reveal a significant discrepancy between the observed rate of periastron advance in the detached eclipsing binary star systems DI Herculis and V541 Cygni and the values theoretically predicted from the combined classical and general relativistic effects. A modification to Newton's gravitational theory is proposed in this investigation to account for these discrepancies, and is represented by

$$\mathbf{F} = -\frac{Gm_1m_2}{r^3}\boldsymbol{r} - \frac{G_o m_1 m_2}{r^2}\boldsymbol{r}$$

where G_o is a second gravitational constant. The two body problem is solved analytically in closed form, resulting in a retrograde contribution to the advance of periastron. Numerical values of G_o were calculated from an analysis of the available data for each of these binary star systems, resulting in a value of $G_o = (1.5 \pm 0.3) \times 10^{-27}\ \mathrm{m^2\,kg^{-1}\,s^{-2}}$ from the analysis of V541 Cygni, and $G_o = (1.5^{+0.3}_{-1.5}) \times 10^{-27}\ \mathrm{m^2\,kg^{-1}\,s^{-2}}$ from an analysis of DI Herculis. The level of agreement between these values supports the assumption that the rotational axes of the stars in V541 Cygni are oriented perpendicular to the orbital plane, as opposed to the highly inclined orbits of the stars observed in DI Herculis. An independently determined value of G_o was calculated through an analysis of the Pioneer anomaly data, resulting in $G_o = (3.00 \pm 0.37) \times 10^{-27}\ \mathrm{m^2\,kg^{-1}\,s^{-2}}$. Within a factor of two, this value of G_o agrees with the results obtained from DI Herculis and V541 Cygni. The proposed theory predicts the "turn on" of the Pioneer anomaly to occur at a heliocentric distance of 10.0 AU, in good agreement with observation.

Keywords Binary stars · Gravitational constant · Pioneer anomaly · DI Herculis · V541 Cygni · Galaxy rotation curves · Modified gravity · Gravity · Astrophysics

1 Introduction

Progress in the field of physics has often been stimulated by the observation of phenomena that could not be adequately explained through the application of existing physical models and theories. With the publication of the *Philosophiae Naturalis Principia Mathematica*, Isaac Newton (Newton 1687) established the theory of gravitational attraction and the laws of motion that bear his name. According to Newton's law of gravity, the force exerted on one body by another due to the gravitational attraction between them is directly proportional to the product of their individual masses and inversely proportional to the square of the distance separating their respective centers of gravity. Using the fundamental laws of motion and gravity, Newton was able to successfully derive Kepler's laws of planetary motion. Several centuries later these same laws would be used to predict the existence of previously undiscovered planets.

Perturbations that were detected in the orbit of the planet Uranus lead Urbain Jean Joseph Le Verrier (and independently John Couch Adams) to predict the existence of the planet Neptune. Based on his analysis Le Verrier was able to approximate the location of this new planet, and in 1846 at the Berlin Observatory Johann Galle observed Neptune orbiting in close proximity to the predicted location. Newton's laws appeared to accurately account for the observed planetary motions, and this success prompted Le Verrier to later

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focus his efforts on the anomalous precession of Mercury's perihelion. In his analysis of the gravitational pull exerted on Mercury by the other planets, Le Verrier concluded that 38 arc seconds could not be accounted for in Mercury's precession (the modern value is 43 arc seconds). He attributed this discrepancy to the existence of a previously unknown mass or masses orbiting the Sun inside the orbit of Mercury. Although early reports indicated that a small planet had been observed (which was later named Vulcan), subsequent attempts to verify these observations failed. This failure of Newton's law of gravity to accurately account for the advance in the precession of Mercury's perihelion as well as it's incompatibility with Special Relativity prompted Albert Einstein to formulate a revolutionary new theory of gravity.

When Einstein published *The Foundation of The General Theory of Relativity* (Einstein 1916) he presented a comprehensive theory of gravity which incorporated Newton's law of gravity as a special limiting case. This new theory was able to accurately account for the missing 43 arc seconds in the precession of Mercury's perihelion advance as well as describe several other aspects of gravitational attraction that were not adequately accounted for by Newton's theory. Even though both Newton's and Einstein's theories have been extraordinarily successful in describing an incredible variety of physical phenomena, there are still well established observations that they do not accurately explain. Examples include the apsidal motion predicted from the combined classical and general relativistic effects in specific detached eclipsing binary star systems, as well as the high rotational velocities of stars observed in the disks of spiral galaxies.

Galaxy rotation curves represent the orbital velocity of massive objects such as stars or dust as a function of their radial distance from the galactic center. And these curves are one of the tools that are used for determining the mass distributions in spiral galaxies. Current studies of spiral galaxy rotation curves reveal a marked discrepancy between the predictions of Newtonian mechanics and the rotational velocities of the objects observed within the disk portions of these galaxies. Typical rotation curves show a steeply rising velocity profile within the central bulge of the galaxy, followed by a relatively constant velocity profile within the disk. This behavior is inconsistent with the predictions of Einstein's General Theory of Relativity and Newton's law of gravity. It has been theorized that these observations betray the existence of dark matter halos (Sofue and Rubin 2001) which are calculated to account for the majority of the mass in some galaxies. Other theoretical explanations such as Modified Newtonian Dynamics (Milgrom 1983a, 1983b) do not invoke the existence of dark matter, but theorize that a modified version of Newton's laws are required to accurately describe the gravitational acceleration that produces the observed behavior in galaxy rotation curves. The purpose of this research is to describe a modification to Newton's original gravitational theory that may be capable of accounting for these unexplained observations. This theory, if independently confirmed, would have profound astrophysical implications regarding the evolution of binary star systems, galaxy formation and evolution, large scale structure, gravitational lensing, and numerous other astrophysical phenomena.

2 Binary star systems

Insight may be gained into the potential physical causes behind the observations obtained from galaxy rotation curves by studying the properties of a less complex system, such as a detached eclipsing binary star system. An example of this two-body system is presented in Fig. 1. It consists of two masses rotating about their common center of mass. If the origin of the coordinate system is fixed in space relative to the center of mass of the binary star system, and the x–y plane is co-planar with the plane of the orbit, then the z axis of this inertial reference frame is oriented perpendicular to the orbital plane. Let the mass of the primary star be m_1 and suppose that it is located at position vector $\boldsymbol{r}_1$ at time $t = 0$. Let the mass of the secondary star be m_2 and suppose it is located at position vector $\boldsymbol{r}_2$ at time $t = 0$. If the primary star exerts a gravitational force $\mathbf{F}$ on the secondary star, then by Newton's third law the secondary star exerts an equal and opposite force of $-\mathbf{F}$ on the primary star. If there are no external forces acting on this system then the fundamental equations of motion for the two-body system are

$$m_1 \frac{d^2\boldsymbol{r}_1}{dt^2} = -\mathbf{F} \tag{1}$$

$$m_2 \frac{d^2\boldsymbol{r}_2}{dt^2} = \mathbf{F} \tag{2}$$

The position vector of the center of mass of this binary system is located at

$$\boldsymbol{r}_{cm} = \frac{m_1\boldsymbol{r}_1 + m_2\boldsymbol{r}_2}{m_1 + m_2} \tag{3}$$

If we define the position vector $\boldsymbol{r}$ such that

$$\boldsymbol{r} = \boldsymbol{r}_2 - \boldsymbol{r}_1 \tag{4}$$

then the position vectors of the primary and secondary stars can be represented by

$$\boldsymbol{r}_1 = \boldsymbol{r}_{cm} - \frac{m_2}{m_1 + m_2}\boldsymbol{r} \tag{5}$$

$$\boldsymbol{r}_2 = \boldsymbol{r}_{cm} + \frac{m_1}{m_1 + m_2}\boldsymbol{r} \tag{6}$$

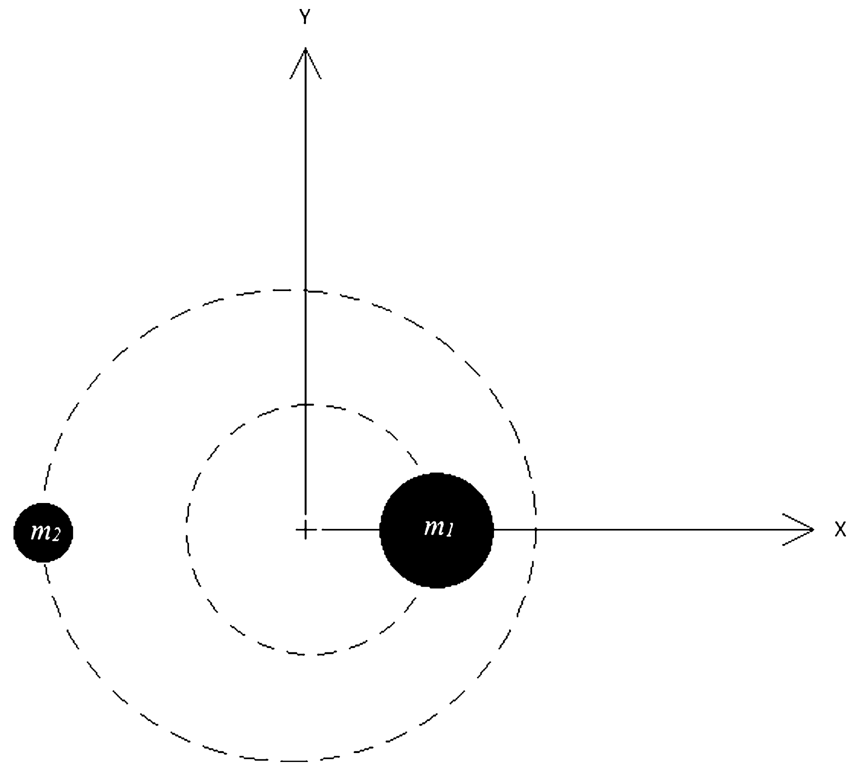

Fig. 1 Schematic diagram of a binary star system. The origin of the coordinate system is fixed in space at the center of mass (+) of the binary star system, with the x–y plane co-planar with the plane of the orbit, and the z axis of this inertial reference frame oriented perpendicular to the page. The mass of the primary star is m_1 and the mass of the secondary star is m_2

Substituting (5) and (6) into (1) and (2), and noting that the center of mass of an isolated system does not accelerate, we find that both of these equations yield

$$\mu \frac{d^2\boldsymbol{r}}{dt^2} = \mathbf{F} \tag{7}$$

where the term μ is the reduced mass and is given by

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \tag{8}$$

The original two-body problem can be reduced to an equivalent one-body problem by suitable choice of the variables. In the equivalent problem, the force $\mathbf{F}$ is the same as that acting on both objects in the original problem (with the exception of a minus sign) and the mass term is represented by the reduced mass.

Detailed analyses by independent research groups over time periods spanning multiple decades reveal a significant, consistent discrepancy between the rates of periastron advance observed in DI Herculis and V541 Cygni and the values theoretically predicted from the combined classical and general relativistic effects (Popper 1982; Guinan and Maloney 1985; Guinan et al. 1996; Sanberg Lacy 1998; Dariush and Riazi 2003; Wolf and Kotkova 2006). The apsidal motion observed in these binary star systems is significantly less than that predicted theoretically, and attempts to account for these discrepancies have encountered varying levels of success (Reisenberger and Guinan 1989; Claret 1998). According to Newton's law of gravity, the gravitational force $\mathbf{F}$ that the primary star exerts on the secondary star is

$$\mathbf{F} = -\frac{G m_1 m_2}{r^3} \boldsymbol{r} \tag{9}$$

The extrapolation of Newton's law to govern the behavior of systems separated by stellar distances is a natural extension of its known range of applicability. But there are compelling reasons to consider alternative gravitational force laws at these large separation distances. Slava Turyshev and Viktor Toth concluded in their publication *The Pioneer Anomaly in Light of New Data* (Turyshev and Toth 2009), "Concluding, we mention that before Pioneer 10 and 11, Newtonian gravity had never been measured—and was therefore never confirmed—with great precision over great distances. The unique 'built-in' navigation capabilities of Pioneer 10 and 11 allowed them to reach the levels of 10^{-10} m/s^2 in acceleration sensitivity. Such an exceptional sensitivity means that Pioneer 10 and 11 represent the largest-scale experiment to test the gravitational inverse square law ever conducted. However, the experiment failed to confirm the validity of this fundamental law of Newtonian gravity in the outer regions of the solar system. One can demonstrate, beyond 15 AU the difference between the predictions of Newton and Einstein is negligible. So, at the moment, two forces seem to be at play in deep space: Newton's laws of gravity and the mysterious Pioneer anomaly. Until the anomaly is thoroughly accounted for by natural causes, and can therefore be eliminated from consideration, the validity of Newton's laws in the outer solar system will remain unconfirmed."

Velocity profiles observed in the disk regions of spiral galaxies by Sofue and Rubin (2001) are also inconsistent with the predictions of the Newtonian gravitational potential (neglecting in this analysis any contribution from theoretical dark matter halos). If dark matter is not invoked, it would appear that there is a fundamental change that occurs in the gravitational force law as the length scale changes from the diameter of our own planetary orbit to that associated with the disk dimensions of a spiral galaxy. Logarithmic corrections to the Newtonian gravitational potential have been investigated by a number of research teams attempting to account for the behavior observed in spiral galaxy rotation curves (Kirillov 2006; Sobouti 2007; Fabris and Campos 2008). For example, Sobouti (2007) provided a theoretical justification for the additional logarithmic term in the gravitational potential function as a part of the framework of the $f(R)$ modifications of General Relativity (R being the Ricci scalar). The addition of a logarithmic corrective term to the Newtonian gravitational potential results in a gravitational force law that can be expressed in the form,

$$\mathbf{F} = -\frac{Gm_1m_2}{r^3}\boldsymbol{r} - \frac{G_o m_1 m_2}{r^2}\boldsymbol{r} \tag{10}$$

where G_o is a second gravitational constant. This modification to Newton's original gravitational theory will be calibrated in this investigation based upon the rate of periastron advance observed in V541 Cygni and DI Herculis, and independently through an analysis of the Pioneer Anomaly data. In this manner, values will be determined for the unknown constant G_o. These calculations will be presented below. Substituting (8) and (10) into (7) and re-arranging we find

$$\frac{m_1m_2}{m_1+m_2}\frac{d^2\boldsymbol{r}}{dt^2} = -\frac{Gm_1m_2}{r^3}\boldsymbol{r} - \frac{G_o m_1 m_2}{r^2}\boldsymbol{r} \tag{11}$$

This reduces to

$$\frac{d^2\boldsymbol{r}}{dt^2} = -\frac{GM}{r^3}\boldsymbol{r} - \frac{G_o M}{r^2}\boldsymbol{r} \tag{12}$$

where

$$M = m_1 + m_2 \tag{13}$$

Expressing (12) in cylindrical polar coordinates

$$(\ddot{r} - r\dot{\theta}^2)\boldsymbol{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\boldsymbol{e}_\theta = -\frac{GM}{r^2}\boldsymbol{e}_r - \frac{G_o M}{r}\boldsymbol{e}_r \tag{14}$$

where the unit vectors $\boldsymbol{e}_r$ and $\boldsymbol{e}_\theta$ are, in terms of their Cartesian x–y coordinates, given by

$$\boldsymbol{e}_r = (\cos\theta, \sin\theta) \tag{15}$$

$$\boldsymbol{e}_\theta = (-\sin\theta, \cos\theta) \tag{16}$$

Since these unit vectors are mutually orthogonal, (14) can be separated into individual components to give the radial equation of motion

$$\ddot{r} - r\dot{\theta}^2 = -\frac{GM}{r^2} - \frac{G_o M}{r} \tag{17}$$

And the tangential equation of motion,

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0 \tag{18}$$

Equation (18) can be equivalently expressed as

$$\frac{d(r^2\dot{\theta})}{dt} = 0 \tag{19}$$

Integrating (19) we find that

$$h = r^2\dot{\theta} \tag{20}$$

where h is the orbital angular momentum per unit mass, a constant of the motion. Let

$$u = \frac{1}{r} \tag{21}$$

Substituting u for $1/r$ from (21) into (17), calculating the expressions for the respective derivatives of u and substituting into the resulting equation and re-arranging,

$$\frac{d^2u}{d\theta^2} + u = \frac{GM}{h^2} + \frac{G_o M}{h^2}\frac{1}{u} \tag{22}$$

Equation (22) can be equivalently expressed as

$$\frac{d\left(\frac{du}{d\theta}\right)}{d\theta}du = \left(-u + \frac{GM}{h^2} + \frac{G_o M}{h^2}\frac{1}{u}\right)du \tag{23}$$

Integrating (23) we find

$$\frac{1}{2}\left(\frac{du}{d\theta}\right)^2 = -\frac{1}{2}u^2 + \frac{GM}{h^2}u + \frac{G_o M}{h^2}\ln u + K \tag{24}$$

where K is an integration constant. Equation (24) may be integrated numerically and the results analyzed to solve for the unknown constant G_o. However, an approximation may be made for the logarithmic term that appears on the right hand side of (24) that allows an analytical, closed form solution to be determined in special limiting cases. This approximation is valid for values of u that fall within the interval represented by the reciprocal of the aphelion and perihelion distances associated with the binary star systems V541 Cygni and DI Herculis. The resulting value of G_o is consistent with the value that is determined numerically from (24).

For both of the binary star systems V541 Cygni and DI Herculis, the $\ln u$ term is five orders of magnitude smaller than each of the quadratic, linear, and constant terms that appear on the right hand side of (24). This may be verified with the value of G_o that will be calculated. The natural logarithm term is therefore much smaller than any of the other individual terms. The intervals over which the $\ln u$ term is evaluated are represented by the reciprocals of the aphelion and perihelion distances for both of these binary star systems, and due to the sizes of these respective intervals the corresponding range of $\ln u$ is quite restricted. The $\ln u$ function is also a much more slowly changing function of u than the sum of the remaining terms, and if this function is approximated by a quadratic curve fit represented by $Au^2 + Bu + C$, then each of the terms in this curve fit (when multiplied by the coefficient G_oM/h^2) can be viewed as a perturbation or small correction to the respective term contained on the right hand side of (24). The constants A, B, and C are determined by requiring that the area beneath the quadratic curve fit equal the area beneath the natural logarithmic function on this interval, and the slope of the curve fit match the slope of the natural logarithmic function at the endpoints of the interval. The resulting curve fit is an excellent approximation for ln u on this interval of u. Making this substitution in (24), rearranging and integrating we find,

$$\int d\theta = -\int \frac{du}{\sqrt{(-1+\frac{2G_oMA}{h^2})u^2+2(\frac{GM}{h^2}+\frac{G_oMB}{h^2})u+2(\frac{G_oMC}{h^2}+K)}} \tag{25}$$

Performing the integration and solving for u,

$$u = E\cos\left((\theta-\theta_o)\sqrt{1-\frac{2G_oM}{h^2}A}\right) + D \tag{26}$$

where the constants E and D are defined by

$$D = \frac{GM + BG_oM}{h^2 - 2G_oMA} \tag{27}$$

$$E^2 - D^2 = \frac{2G_oMC + 2h^2K}{h^2 - 2G_oMA} \tag{28}$$

Without any loss of generality we can set $\theta_o = 0$ by rotating the coordinate system about the z-axis. Equation (26) then reduces to,

$$u = E\cos\left(\theta\sqrt{1-\frac{2G_oMA}{h^2}}\right) + D \tag{29}$$

And we find,

$$r = \frac{1}{E\cos(\theta\sqrt{1-\frac{2G_oMA}{h^2}}) + D} \tag{30}$$

Note that in the limit where G_o approaches zero, the solution for r expressed in (30) reduces to the well known solution for Newtonian gravity, as it should. If we set

$$\varepsilon = 1 - \sqrt{1-\frac{2G_oMA}{h^2}} \tag{31}$$

then the period P in radians per orbit is given by

$$P = \frac{2\pi}{1-\varepsilon} \tag{32}$$

For $\varepsilon \ll 1$, the period is approximately

$$P = 2\pi(1+\varepsilon) \tag{33}$$

The periastron precession per orbit δ is obtained by subtracting 2π from the period. This results in the following expression for the orbital precession in units of radians per orbit,

$$\delta = 2\pi\varepsilon \tag{34}$$

Substituting for ε from (31) into (34), and converting to units of degrees per orbit,

$$\delta = 360\left(1-\sqrt{1-\frac{2G_oMA}{h^2}}\right) \tag{35}$$

Thus, the second term appearing in (10) impacts the apsidal motion of a binary star system in accord with (35). Note the parameter A that appears in this equation is equal to one half of the average curvature of the natural logarithmic function over the representative orbital interval, so that the value of δ is always negative. The contribution to the resulting precession is therefore retrograde. The magnitude of this effect upon the rate of apsidal motion, represented here by $\dot{\omega}_{cont}$, is determined by multiplying δ by the number of orbits completed per century N such that

$$\dot{\omega}_{cont} = N\delta \tag{36}$$

Table 1 lists the relevant orbital parameters of DI Herculis and V541 Cygni. The differences between the observed rates of apsidal motion $\dot{\omega}_{obs}$ and the corresponding rates of apsidal motion predicted from the combined classical and general relativistic effects $\dot{\omega}_{th}$ appear on the right hand side of this table under the designation $(\dot{\omega}_{obs} - \dot{\omega}_{th})$. Note that the predicted apsidal motion rates listed in Table 1 exceed the observed rates for both of these binary star systems.

DI Herculis consists of two main-sequence B stars (B4 V and B5 V) moving in a highly eccentric orbit ($e = 0.489$). This system has a high orbital inclination ($i = 89.3°$) and small fractional radii of the stars ($R_1/a = 0.0621$ and $R_2/a = 0.0574$), and produces very deep and narrow eclipses. This allows accurate timings of minimum light, which permits very accurate apsidal motion studies to be

Table 1 Orbital, physical, and calculated parameters for V541 Cygni and DI Herculis. The values that appear in this table for V541 Cygni were derived from the data contained in the reference Sanberg Lacy (1998). For DI Herculis, the values were derived from the references Gunian and Maloney (1985) and Albrecht et al. (2009)

Binary system	Semi-major axis	Total mass	N (Orbits/100 yr)	A (m^2)	$\dot{\omega}_{obs} - \dot{\omega}_{th}$
V541 Cygni	42.7 R_{sun}	4.48 M_{sun}	2381	-3.40×10^{20}	$-0.29°/100$ yr
DI Herculis	43.2 R_{sun}	9.67 M_{sun}	3462	-3.43×10^{20}	$-0.42°/100$ yr

conducted. Since its discovery by Hoffmeister (1930), this system has been widely studied. While the effects of general relativity are predicted to dominate the rate of apsidal advance in this system, studies indicated that there was a significant discrepancy between the theoretical predictions and observations (Popper 1982; Guinan and Maloney 1985; Dariush and Riazi 2003). Previous observations of DI Herculis by Guinan and Maloney (1985) resulted in a value for the rate of periastron advance that was approximately one seventh the value expected from the combined classical Newtonian and general relativistic effects. This result led the authors to conclude that either an undiscovered or misunderstood classical effect was present in this system, or that there was a significant problem with the predictions of general relativity. Many theories have subsequently been proposed to account for this discrepancy (Reisenberger and Guinan 1989; Claret 1998; Moffat 2004; Hsuan and Mardling 2006; Kozyreva and Bagaev 2009).

An inherent assumption in this analysis is that the rotational axes of the stars are oriented perpendicular to the orbital plane. Shakura suggested that the discrepancy between theory and observation could be resolved if the rotational axes of the stars were highly inclined to the orbital plane (Shakura 1985), and observations that have been reported recently support this hypothesis (Albrecht et al. 2009). In their analysis, Albrecht et al. found that the orbital axes of the primary and secondary stars are inclined at angles of 72° and 84° respectively to the orbital plane. The observed precession rate presented in their study of $(1.04° \pm .15°)/100$ yr was quoted from the literature, while the Monte Carlo calculation that was performed resulted in a net theoretical rate of $1.46°/100$ yr. The difference between their theoretical and observed rates appears in Table 1 as $-0.42°/100$ yr. By equating $\dot{\omega}_{cont}$ in (36) to this value for DI Herculis, the value of the gravitational constant G_o can be determined. Making this substitution into (36) and substituting for δ from (35) into (36), re-arranging and ignoring the second order terms,

$$G_o = \frac{h^2(\dot{\omega}_{obs} - \dot{\omega}_{th})}{360AMN} \qquad (37)$$

Utilizing the data from Table 1 we find that $G_o = (1.5^{+0.3}_{-1.5}) \times 10^{-27}$ m^2 kg^{-1} s^{-2}. Note that the relatively large error interval generated from the Monte Carlo calculation results in a lower limit of zero for G_o.

V541 Cygni consists of a pair of detached B9.5 V stars moving in a highly eccentric orbit ($e = 0.479$) with a very high orbital inclination ($i = 89.9°$). Previous studies of this system (Guinan et al. 1996; Sanberg Lacy 1998; Wolf and Kotkova 2006) have revealed that the effects of general relativity are predicted to dominate the rate of apsidal advance. But the observed rate of apsidal advance is also significantly less than that predicted theoretically. Using the data from Table 1 in (37), and utilizing the inherent assumption that the rotational axes of the stars are oriented perpendicular to the orbital plane, we find that $G_o = (1.5 \pm 0.3) \times 10^{-27}$ m^2 kg^{-1} s^{-2}. This value of G_o matches the value determined from the analysis of DI Herculis to two significant figures. This level of agreement strongly supports the assumption that the rotational axes of the stars in V541 Cygni are in fact oriented perpendicular to the orbital plane, as opposed to the highly inclined orbits of the stars observed in DI Herculis.

3 The Pioneer anomaly

The Pioneer 10 spacecraft was designed as a deep space probe intended to gather information on the asteroid belt and to perform direct observations of the planet Jupiter. It was launched in March of 1972, and Pioneer 11 was launched soon thereafter in April of 1973. Although they are two of the most precisely navigated spacecraft ever launched, both experienced a small, anomalous drift in their 2-way tracking signals at heliocentric distances between approximately 20 and 70 AU from the Sun. This blue shifted Doppler frequency drift is most frequently represented as a constant acceleration of $(8.74 \pm 1.33) \times 10^{-10}$ m/s^2 directed toward the Sun (Anderson et al. 2002). This unexplained acceleration is commonly known as the Pioneer anomaly, and numerous attempts have been made to explain the source of this acceleration (Anderson et al. 2002; Lammerzahl et al. 2008). These explanations range from an exhaustive list of potential systematic sources such as non-isotropic radiative cooling of the spacecraft, gas leakage from the spacecraft's propulsive system, and differential emissivity of the radioisotope thermal generators, to more exotic theoretical explanations such as drag created from interaction with interplanetary and interstellar dust, solar radiation and solar wind pressure, and gravitational forces from dark matter. No satisfactory explanation has yet been found. The general consensus among physicists is that the Pioneer anomaly likely stems from an unidentified systematic source (Scheffer 2003). However there is also the possibility that it results from a more fundamental gravitational origin that has remained unidentified. The second possibility is the perspective adopted in this paper, and the data from the Pioneer Anomaly is analyzed accordingly.

Numerical values of the unmodeled accelerations experienced by the Pioneer spacecraft were originally published by Nieto and Anderson (2005), and are represented by a_u in Table 2. A review of the relevant data suggests that the anomalous acceleration of the Pioneer 11 spacecraft is not a constant function of heliocentric distance r, since this data exhibits a distinct variation with respect to r during the initial stages of the observations. There is an apparent "turn on" of the anomaly in the vicinity of Saturn encounter, followed by a relatively steep rise in a_u over the next four years. The anomalous acceleration for this spacecraft peaks at $r = 18.90$ AU, and following this peak a very gradual decreasing trend is noted in a_u overall for heliocentric distances up to 29.50 AU. If the data from the latter part of this curve is combined with the relevant Pioneer 10 data and the anomalous acceleration is approximated as a constant function of r, then the resulting acceleration value that is obtained is $(8.74 \pm 1.33) \times 10^{-10}$ m/s^2. While this is the value most often quoted to represent the Pioneer anomaly, the model proposed in this investigation can provide a much more accurate fit to the experimental data.

Table 2 The unmodeled accelerations (a_u) experienced by the Pioneer 10 and Pioneer 11 spacecraft, and corresponding values of the heliocentric distance (r). The values presented in this table were derived from the reference Nieto and Anderson (2005)

r (10^{12} m)	a_u (10^{-10} m s^{-2})
Pioneer 11 Data	
1.405	1.56
1.819	6.28
2.094	8.05
2.518	8.15
2.827	9.03
3.329	8.13
3.486	8.98
3.979	8.56
4.413	8.33
Pioneer 10 Data	
3.943	8.68
4.320	8.88
4.733	8.59
5.137	8.43
5.323	7.67
5.584	8.43
6.072	7.45
6.463	8.09
6.837	8.24

For the purposes of the current analysis, it is assumed that (10) represents the applicable gravitational force law within the confines of our solar system. Given this assumption, the unmodeled acceleration experienced by the Pioneer spacecraft may be attributed to the difference between the original predictions of Newton's law of gravity, as calculated by the members of the NASA team, and the predictions of (10). To a first approximation we find,

$$m_2 a_u = \left[-\frac{1.00000565k^2 m_2}{r^2}\right] - \left[-\frac{(GM)_c m_2}{r^2} - \frac{G_o M m_2}{r}\right] \quad (38)$$

where m_2 is the mass of the Pioneer spacecraft, and $1.00000565k^2$ is the value of GM used by Nieto and Anderson to represent the central mass at the solar system barycenter for the Pioneer spacecraft where k is Gauss' constant, and M represents the solar mass supplemented by the masses of Mercury, the Earth, the Moon, and Venus. Note that the assumption that (10) is the applicable gravitational force law results in a fundamental change in the numerical value of GM. This change arises due to the presence of the second term in (10). The corrected value of GM is represented by the quantity $(GM)_c$ that appears in (38). The difference between this value and the value used by Nieto and Anderson will be defined as $\Delta(GM)$ where

$$\Delta(GM) = (GM)_c - 1.00000565k^2 \quad (39)$$

Substituting (39) into (38), dividing through by m_2 and multiplying by r,

$$a_u r = \frac{\Delta(GM)}{r} + G_o M \quad (40)$$

A least squares curve fitting technique can be applied to the Pioneer anomaly data through (40) to determine the values for $\Delta(GM)$ and $G_o M$ by treating the quantity $a_u r$ as the dependent variable and $1/r$ as the independent variable. Using the methods of calculus, it can be easily shown that the least squares approach results in the following expressions for $\Delta(GM)$ and $G_o M$,

$$\Delta(GM) = \frac{18[\sum_{i=1}^{18}(a_u)_i] - [\sum_{i=1}^{18}(\frac{1}{r_i})][\sum_{i=1}^{18}(a_u r)_i]}{18[\sum_{i=1}^{18}(\frac{1}{r_i})^2] - [\sum_{i=1}^{18}(\frac{1}{r_i})]^2} \quad (41)$$

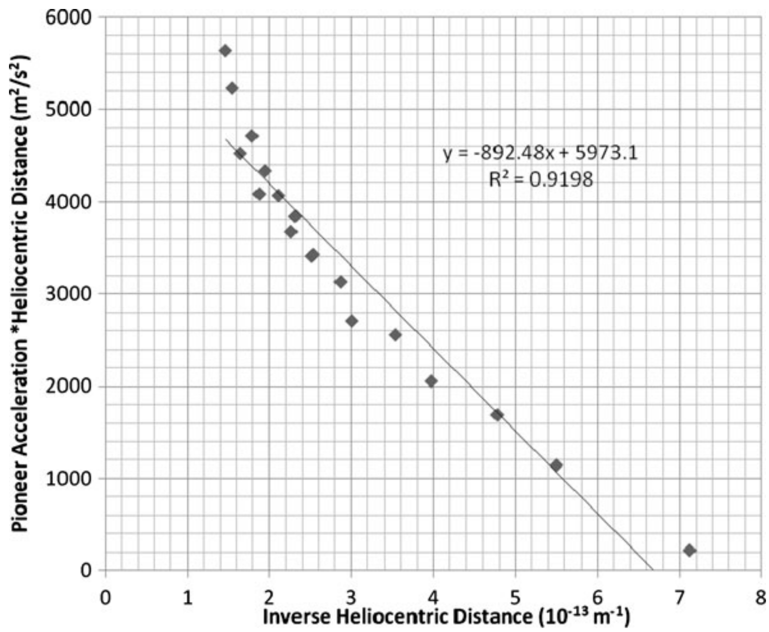

Fig. 2 The Pioneer anomaly data expressed as $a_u r$ versus $1/r$. The curve fit of (40) is represented by the straight line through the data. The "turn on" of the anomaly is calculated to occur at $r = 1.49 \times 10^{12}$ m, in good agreement with observation

$$G_o M = \frac{[\sum_{i=1}^{18}(a_u r)_i][\sum_{i=1}^{18}(\frac{1}{r_i})^2] - [\sum_{i=1}^{18}(\frac{1}{r_i})][\sum_{i=1}^{18}(a_u)_i]}{18[\sum_{i=1}^{18}(\frac{1}{r_i})^2] - [\sum_{i=1}^{18}(\frac{1}{r_i})]^2} \tag{42}$$

Table 2 lists the unmodeled accelerations a_u experienced by the Pioneer 10 and Pioneer 11 spacecraft with corresponding values of heliocentric distance r ranging from 9.39 AU to 45.70 AU. Substituting the data from Table 2 into (41) and (42), we find that $G_o M = 5.97 \times 10^3$ m^2 s^{-2} and $\Delta(GM) = -8.92 \times 10^{15}$ m^3 s^{-2}. A graphical representation of the resulting least squares curve fit is shown in Fig. 2. The x-intercept represents the point at which the anomalous acceleration falls to zero, and the "turn on" of the Pioneer anomaly is initiated. According to Fig. 2, the "turn on" of the Pioneer anomaly is calculated to occur at $r = 1.49 \times 10^{12}$ m, or 10.0 AU. This prediction is in good agreement with observation.

Substituting the value determined for $\Delta(GM)$ into (39) we find that $(GM)_c = 1.327035 \times 10^{20}$ m^3 s^{-2}. Taking the ratio of $G_o M$ to $(GM)_c$ the mass term cancels out and we find that $G_o/G = 4.50 \times 10^{-17}$ m^{-1}. Using the 2006 CODATA-recommended value of the gravitational constant G of $(6.67428 \pm 0.00067) \times 10^{-11}$ m^3 kg^{-1} s^{-2} in this ratio, the resulting value for the second gravitational constant is $G_o = (3.00 \pm 0.37) \times 10^{-27}$ m^2 kg^{-1} s^{-2}. Within a factor of two, the value calculated for G_o from the Pioneer anomaly data agrees with the values determined from the analyses of V541 Cygni and DI Herculis. This suggests that the Pioneer anomaly may also stem from the same gravitational origin that was presented in this investigation to explain the rate of periastron advance observed in these binary star systems.

4 Conclusions

A modification to Newton's gravitational theory is proposed in this investigation to quantitatively account for the discrepancies that have been identified between theory and observation in the rate of periastron advance in the detached eclipsing binary star systems V541 Cygni and DI Herculis. An analytical, closed form solution was derived for the two body problem resulting in a retrograde contribution to the rate of periastron advance. Numerical values of G_o were calculated from an analysis of the available data for each of these systems, resulting in a value of $G_o = (1.5 \pm 0.3) \times 10^{-27}$ m^2 kg^{-1} s^{-2} from the analysis of V541 Cygni, and $G_o = (1.5^{+0.3}_{-1.5}) \times 10^{-27}$ m^2 kg^{-1} s^{-2} from an analysis of DI Herculis. The level of agreement between these values strongly supports the assumption that the rotational axes of the stars in V541 Cygni are oriented perpendicular to the orbital plane, as opposed to the highly inclined orbits of the stars observed in DI Herculis.

An independently determined value for the second gravitational constant was calculated from an analysis of the Pioneer anomaly data, resulting in a value of $G_o = (3.00 \pm 0.37) \times 10^{-27}$ m^2 kg^{-1} s^{-2}. Within a factor of two, this value of G_o agrees with the results obtained from DI Herculis and V541 Cygni. The level of agreement between the values of G_o calculated over the separation distances associated with the binary star systems that were analyzed (0.2 AU) and the separation distances associated with the Pioneer anomaly (up to 45.70 AU) suggests that the Pioneer anomaly may stem from the same gravitational origin that was presented in this investigation to explain the rate of periastron advance observed in these binary star systems.

The least squares curve fit to the Pioneer data resulted in a prediction for the "turn on" of the Pioneer anomaly at a heliocentric distance of 1.49×10^{12} m, or 10.0 AU. This result is in good agreement with observation.

5 Discussion

The impact of adding a logarithmic corrective term to the Newtonian gravitational potential function was evaluated in this investigation by examining the effect of the resulting gravitational force law upon the Pioneer anomaly and upon the behavior of the binary star systems V541 Cygni and DI Herculis. Both of these binary systems have been studied for decades, and each possesses the characteristics and properties that make them appropriate candidates for this analysis. While the contributions of general relativity were correctly predicted to dominate the rate of apsidal advance in each of these binary systems, the classical Newtonian contribution appears to be overstated. Correction of the classical contribution through the analysis provided herein more accurately predicts the rate of periastron advance observed in both of these binary star systems.

The origin of the anomalous acceleration experienced by the Pioneer spacecraft has remained a topic of intense debate. The predominant point of view expressed in the literature regarding the Pioneer anomaly is that the anomalous acceleration is caused by a force that acts in addition to the Newtonian gravitational force, whether this additional force is systematic in origin or stems from some other unidentified source. The view adopted in this investigation is that the Pioneer anomaly stems from the *difference* between the predictions of Newton's law of gravity and the predictions of the gravitational force law represented by (10). If correct, this would require the anomalous acceleration to be a function of heliocentric distance r. However, the anomalous Pioneer acceleration is overwhelmingly quoted in the literature as a constant acceleration that is independent of r. But this is likely to change soon.

Turyshev and Toth have announced that they are currently in the process of performing an exhaustive evaluation of the extended set of radiometric Doppler data recently acquired for the Pioneer spacecraft. And in recent interviews they have made statements indicating that the preliminary analysis of the extended Pioneer data suggests that the anomalous acceleration is not a constant value as previously believed, but is a decreasing function of time, and therefore a decreasing function of r. In the conclusions section of their most recent publication on the Pioneer anomaly (Turyshev and Toth 2010), they state that several independent research studies suggest that the generation and dissipation of heat by the radioisotope thermal generators may have provided a greater contribution to the anomalous acceleration of the Pioneer spacecraft than was previously estimated. If this is verified, then the acceleration values determined by Nieto and Anderson (2005) contained in Table 2 of this paper are overstated to some degree. By correcting these values accordingly, the resulting value determined for G_o from the Pioneer acceleration data should be lower than the value that was calculated in the present study. The correction of the Pioneer anomaly data, if warranted, should therefore bring this value of G_o in closer agreement with the values determined from the analysis of V541 Cygni and DI Herculis.

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